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STABILIZATION OF DYNAMIC SYSTEMS USING POSITIONAL SOLUTIONS OF SPECIAL OPTIMAL CONTROL PROBLEMS†

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An algorithm of the operation of a stabilizer of dynamic systems is described. The stabilizer is based on the use of positional solutions of the problem of optimal control with a mixed quality criterion. Copyright © 1996 Elsevier Science Ltd.

In optimal-control theory there are a number of generally accepted quality criteria, in addition to abstract criteria of general form, which have a clear physical meaning [1]: speed of response, intensity, fuel consumption, a minimum of energy, and so on. The positional solution of special optimal-control problems with these quality criteria can be used [2–3] to design dynamic-system stabilizers.

In this paper we investigate the possibility of using positional solutions of an optimal-control problem with a mixed quality criterion, made up of two of the above-mentioned quality criteria of a particular type.

A classical example of an optimal-control problem with a mixed quality criterion is the Letov–Kalman problem on the analytic construction of an optimal controller. Applications of the Letov–Kalman problem to the stabilization of dynamic systems are well known [4-6]. Success always depended on the possibility of constructing positional solutions.

1. THE STABILIZATION PROBLEM AND THE ACCOMPANYING OPTIMAL-CONTROL PROBLEM

The classical formulation of the problem of stabilizing dynamic systems is as follows. Consider the dynamic system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u, \quad \mathbf{x}(0) \in X_0 \tag{1.1}$$

where X_0 is a certain neighbourhood of the origin of coordinates and $\mathbf{x} \in R^n$. It is required to construct a function $u = u(x)$ so that for any $\mathbf{x}_0 \in \mathbf{X}_0$ the system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u(x), \quad \mathbf{x}(0) = \mathbf{x}_0$$

has a solution and it is asymptotically stable [4].

It was suggested in [7] that positional solutions of optimal-control problems with certain specific quality criteria should be used to stabilize dynamic systems. Many other quality criteria can obviously also be used.

To stabilize system (1.1) we will use the following optimal-control problem with a mixed quality criterion

$$\begin{gathered}\alpha t^* + (1-\alpha) \max_{t \in T(t^*)} |u(t)| \to \min \\ \mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{b}u, \quad \mathbf{x}(0) = \mathbf{x}_0, \quad \mathbf{x}(t^*) = 0, \quad |u(t)| \leqslant L \\ t \in T(t^*) = [0, t^*], \quad (\mathbf{x} \in R^n, \quad u \in R, \quad 0 < \alpha < 1)\end{gathered} \tag{1.2}$$

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in the class of piecewise-continuous functions $u(t)$, $t \in T(t^*)$.

When $\alpha = 1$ problem (1.2) becomes the classical optimal speed-of-response problem. If we put $\alpha = 0$, fix $t^* > 0$ and assume that L is a sufficiently large number, we obtain the problem of constructing an admissible control of minimum intensity.

We will confine ourselves to initial states x_0 for which the optimal open-loop control $u^0(t)$, $t \in T(t^{*0})$, of problem (1.2) satisfies the inequality $|u^0(t)| < L$, $t \in T(t^{*0})$.

We will first formulate the necessary condition for the open-loop control to be optimal. Suppose $u^0(t)$, $t \in T(t^{*0})$, is the solution of problem (1.2). A vector $\mathbf{y} \in R^n$ then exists such that

$$\mathbf{y}'\mathbf{F}(t^{*0},0)\,\mathbf{x}_0 = (1-\alpha)\max_{t \in T(t^{*0})} |u^0(t)| = (1-\alpha)\rho^0$$

$$\mathbf{y}'\mathbf{b}u^0(t^{*0}-0) = -\alpha$$

and along with the solution of the conjugate system

$$\dot{\psi} = -\mathbf{A}'\psi, \quad \psi(t^{*0}) = -\mathbf{y}$$

and the control $u^0(t)$, $t \in T(t^{*0})$, the condition for a maximum is satisfied, namely

$$\psi'(t)\mathbf{b}u^0(t) = \max_{|u| \leqslant \rho^0} \psi'(t)\mathbf{b}u, \quad t \in T(t^{*0}).$$

Notes. 1. We can split problem (1.2) into two problems

$$f(t^*) = \alpha t^* + (1-\alpha)\rho(t^*) \to \min_{0<t^*} \tag{1.3}$$

$$\rho(t^*) = \min_{u(\cdot),\rho} \rho, \quad \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u, \quad \mathbf{x}(0) = \mathbf{x}_0, \quad \mathbf{x}(t^*) = 0 \tag{1.4}$$

$$|u(t)| \leqslant \rho, \quad t \in T(t^*)$$

The second of these is the linear optimal-control problem (the problem of constructing an admissible control of minimum intensity), and the first is the problem of minimizing a function of one variable.

2. The function $f(t^*)$, $t^* > 0$ possesses the following properties: $f(t^*) \to \infty$ as $t^* \to +0$, $f(t^*) \to \infty$ as $t^* \to +\infty$. In general, the function $f(t^*)$, $t^* > 0$ is non-smooth and multiextremal.

3. The necessary conditions for optimality formulated above can be treated as the necessary first-order conditions for a minimum of the function $f(t^*)$, $t^* > 0$.

Henceforth, for simplicity, we will assume that $\alpha = 0.5$.

2. THE POSITIONAL SOLUTION OF THE ACCOMPANYING OPTIMAL-CONTROL PROBLEM AND ITS REALIZATION

By analogy with the previous discussion [7], to stabilize system (1.1) we will use the positional solution of problem (1.2) we will embed it in the family of problems

$$t^* + \rho \to \min, \quad \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u, \quad \mathbf{x}(0) = \mathbf{z}$$

$$\mathbf{x}(t^*) = 0, \quad |u(t)| \leqslant \rho, \quad t \in [0, t^*] \tag{2.1}$$

which depends on the vector $\mathbf{z} \in R^n$. We will denote the open-loop solution of problem (2.1) by $u(t \mid \mathbf{z}$, $t \in [0, t^*(\mathbf{z})]$, $\rho(\mathbf{z})$.

By the necessary conditions of optimality, a vector $\mathbf{y}(\mathbf{z}) \in R^n$ exists (the vector of Lagrange multipliers) such that

$$\mathbf{y}'(\mathbf{z})F(t^*(\mathbf{z}),0)\mathbf{z} = \rho(\mathbf{z}), \quad \mathbf{y}'(\mathbf{z})\mathbf{b}u(t^*(\mathbf{z})-0|\mathbf{z}) = -1$$

and along the solution $\psi_{\mathbf{z}}(t)$, $t \in [0, t^*(\mathbf{z})] = T(\mathbf{z})$ of the conjugate system

$$\dot{\psi} = -\mathbf{A}'\psi, \quad \psi(t^*(\mathbf{z})) = -\mathbf{y}(\mathbf{z}) \tag{2.2}$$

the condition for a maximum is satisfied, namely

$$\psi'_{\mathbf{z}}(t)\mathbf{b}u(t|\mathbf{z}) = \max_{|u|<\rho(\mathbf{z})} \psi'_{\mathbf{z}}(t)\mathbf{b}u, \quad t \in T(\mathbf{z}) \tag{2.3}$$

Note that problem (2.1) can have several optimal open-loop solutions

$$u_i(t|\mathbf{z}), \quad t \in [0, t_i^*(\mathbf{z})]; \quad \rho_i(\mathbf{z}), \quad i = \overline{1,l}, \quad l \geq 1$$

$$t_i^*(\mathbf{z}) + \rho_i(\mathbf{z}) = t_{i+1}^*(\mathbf{z}) + \rho_{i+1}(\mathbf{z}), \quad \rho_i(\mathbf{z}) < \rho_{i+1}(\mathbf{z}), \quad i = \overline{1, l-1}$$

We will agree that we mean by a solution of problem (2.1) the following

$$u(t|\mathbf{z}) = u_1(t|\mathbf{z}), \quad t \in [0, t_1^*(\mathbf{z})], \quad \rho(\mathbf{z}) = \rho_1(\mathbf{z}), \quad t^*(\mathbf{z}) = t_1^*(\mathbf{z})$$

i.e. the optimal control of minimum intensity.

Definition. We will call the function

$$u(\mathbf{z}) = u(+0|\mathbf{z}), \quad \mathbf{z} \in Z \tag{2.4}$$

the (optimal) positional solution of problem (1.2). Here Z is the set of all $\mathbf{z} \in R^n$ for which problem (2.1) has a solution.

We will assume that the function $u(\mathbf{z})$, $\mathbf{z} \in Z$ has been constructed. This closes the initial system. As a result we obtain

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u(\mathbf{x}), \quad \mathbf{x}(0) = \mathbf{z} \tag{2.5}$$

We put

$$\begin{aligned}
&\{t_j(\mathbf{z}), \; j = \overline{1, p(\mathbf{z})}\} = \{t \in T(\mathbf{z}): \Delta_{\mathbf{z}}(t) = 0\} \\
&t_j(\mathbf{z}) < t_{j+1}(\mathbf{z}), \; j = \overline{1, p(\mathbf{z}) - 1}; \; t_0(\mathbf{z}) = 0, \; t_{p(\mathbf{z})+1}(z) = t^*(\mathbf{z}) \\
&k_j(z) = \operatorname{sign} \Delta_{\mathbf{z}}(t), \; t \in]t_j(\mathbf{z}), t_{j+1}(\mathbf{z})[, \; j = \overline{0, p(\mathbf{z})} \\
&\Delta_{\mathbf{z}}(t) = \psi'_{\mathbf{z}}(t)\mathbf{b}, \; t \in T(\mathbf{z})
\end{aligned} \tag{2.6}$$

It follows from the maximum principle (2.3) that

$$u(t|\mathbf{z}) = k_j(\mathbf{z})\rho(z), \quad t \in]t_j(\mathbf{z}), t_{j+1}(\mathbf{z})[, \quad j = \overline{0, p(\mathbf{z})}$$

and we have the relations

$$\mathbf{F}(t^*(\mathbf{z}), 0)\mathbf{z} + \sum_{j=0}^{p(\mathbf{z})} k_j(\mathbf{z})\rho(\mathbf{z}) \int_{t_j(\mathbf{z})}^{t_{j+1}(\mathbf{z})} \mathbf{F}(t^*(\mathbf{z}), t)b\,dt = 0 \tag{2.7}$$

$$\begin{aligned}
&\mathbf{y}'(\mathbf{z})\mathbf{F}(t^*(\mathbf{z}), \; t_j(\mathbf{z}))b = 0, \; j = \overline{1, p(\mathbf{z})} \\
&\mathbf{y}'(\mathbf{z})\mathbf{F}(t^*(\mathbf{z}), 0)\mathbf{z} = \rho(\mathbf{z}), \; \mathbf{y}'(\mathbf{z})\mathbf{b}k_{p(\mathbf{z})}(\mathbf{z})\rho(\mathbf{z}) = -1
\end{aligned} \tag{2.8}$$

For any $z \in Z$ the following relations are obviously true

$$k_j(\mathbf{z}) = \mp 1, \quad j = \overline{0, p(\mathbf{z})}, \quad p(\mathbf{z}) \in \{0, 1, 2, \ldots\}$$

Considerable system (2.5). We will assume that it has a solution $\mathbf{x}_{\mathbf{z}}(\tau) = \mathbf{x}(\tau)$, $\tau \geq 0$, and that this solution is such that the functions $k_j(\tau) = k_j(\mathbf{x}(\tau)), j = \overline{0, p(\tau)}, p(\tau) = p(\mathbf{x}(\tau)), \tau \geq 0$ are piecewise-constant.

It can be shown that in this case the following relation is satisfied for any τ

$$d(t^*(\mathbf{x}(\tau + 0)) + \rho(\mathbf{x}(\tau + 0)))/d\tau = -1 \tag{2.9}$$

In fact, we put

$$p = p(s) = p(\mathbf{x}(s)),\ \ k_j = k_j(s) = k_j(\mathbf{x}(s)),\ \ j = \overline{0,p},\ \ s \in T^+(\tau)$$

where $T^+(\tau)$ is a fairly small right-sided neighbourhood of the point τ. Equation (2.7) then takes the form

$$F(t^*(\mathbf{x}(\tau)),0)\mathbf{x}(\tau) + \sum_{j=0}^{p} k_j\rho(\mathbf{x}(\tau)) \int_{t_j(\mathbf{x}(\tau))}^{t_{j+1}(\mathbf{x}(\tau))} F(t^*(\mathbf{x}(\tau)),t)b\,dt \equiv 0 \tag{2.10}$$

From (2.10) we obtain

$$-\frac{\mathbf{F}(t^*(\mathbf{x}(\tau)),0)\mathbf{x}(\tau)}{\rho(\mathbf{x}(\tau))}\frac{d\rho(\mathbf{x}(\tau))}{d\tau} + \sum_{j=1}^{p}\rho(\mathbf{x}(\tau))(k_j - k_{j-1})\times$$

$$\times\mathbf{F}(t^*(\mathbf{x}(\tau)),\ t_j(\mathbf{x}(\tau)))b\,\frac{dt_j(\mathbf{x}(\tau))}{d\tau} + k_p\rho(\mathbf{x}(\tau))\,\frac{dt^*(\mathbf{x}(\tau))}{d\tau} +$$

$$+\mathbf{F}(t^*(\mathbf{x}(\tau)),0)\mathbf{A}\mathbf{x}(\tau) + \mathbf{F}(t^*(\mathbf{x}(\tau)),0)bk_0\rho(\mathbf{x}(\tau)) = 0 \tag{2.11}$$

Using (2.10) it can be proved that

$$k_p\rho\mathbf{b} = \mathbf{F}(t^*,0)\mathbf{A}\mathbf{x}(\tau) + k_0\rho\mathbf{F}(t^*,0)b + \sum_{j=1}^{p}\rho(k_j - k_{j-1})\mathbf{F}(t^*,t_j)\mathbf{b} \tag{2.12}$$

$$(\rho = \rho(\mathbf{x}(\tau)),\ \ t^* = t^*(\mathbf{x}(\tau)),\ \ t_j = t_j(\mathbf{x}(\tau)),\ \ j = \overline{1,p})$$

We multiply the right- and left-hand sides of (2.12) by $\mathbf{y}'(\tau)$ and use (2.8). We obtain

$$\mathbf{y}'(\tau)(\mathbf{F}(t^*,0)\mathbf{A}\mathbf{x}(\tau) + k_0\rho\mathbf{F}(t^*,0)b) = -1 \tag{2.13}$$

We multiply (2.11) by $\mathbf{y}'(\tau)$ and use (2.8) and (2.13). This gives

$$-d\rho(\mathbf{x}(\tau))/d\tau - dt^*(\mathbf{x}(\tau))/d\tau - 1 = 0$$

Equation (2.9) follows from the last relation.

Relation (2.9) holds for any $\tau \geqslant 0$ and any $\mathbf{z} \in Z$. Hence we see that system (2.5) is asymptotically stable. In fact, if $\mathbf{x}(0) = \mathbf{z}$, the trajectory $x(\tau)$, $\tau \geqslant 0$, of system (2.5) falls on the origin of coordinates after a time t^*, which does not exceed $t^*(\mathbf{z}) + \rho(\mathbf{z})$.

3. AN ALGORITHM FOR CONSTRUCTING A STABILIZING CONTROL. GOVERNING EQUATIONS

The problem of constructing the function $u(\mathbf{x})$, $\mathbf{x} \in Z$ is complicated and has not been realized in practice. However, when stabilizing system (2.5) with a specified (arbitrary) initial state $\mathbf{x}(0) = \mathbf{z}^*$ there is no need to know every function $u(\mathbf{x})$, $\mathbf{x} \in Z$. In fact, suppose the initial state $\mathbf{z}^*$ is specified. It generates the trajectory $\mathbf{z}^*(t)$, $t \geqslant 0$, of system (2.5). Consequently, in this specific process of stabilization we will not use every feedback (2.4) but only its form $u^*(\tau) = u(\mathbf{z}^*(\tau))$, $\tau \geqslant 0$ along the curve $\mathbf{z}^*(\tau)$, $\tau \geqslant 0$. Here there is no need to know the value of the control $u^*(\tau)$ in advance (up to the start of the stabilization process). It will only be necessary at the current instant τ when system (2.5) is in the state $\mathbf{z}^*(\tau)$.

These facts enable us to develop an algorithm for constructing the function $u^*(\tau)$, $\tau \geqslant 0$, in real time for each specific process of stabilizing the state $\mathbf{x}(0) = \mathbf{z}^*$.

We will write this algorithm. We construct the open-loop control $u(t \mid \mathbf{z}^*)$, $t \in T(\mathbf{z}^*)$, $t^*(\mathbf{z}^*)$, $\rho(\mathbf{z}^*)$ and the corresponding optimal vector of the Lagrange multipliers $\mathbf{y}(\mathbf{z}^*) \in R^n$ for problem (2.1) with $\mathbf{z} = \mathbf{z}^*$.

We will denote the parameters (2.6) by $t_j(\mathbf{z}^*), j = \overline{1, p(\mathbf{z}^*)}$; $\rho(\mathbf{z}^*)$, $k_j(\mathbf{z}^*), j = \overline{0, p(\mathbf{z}^*)}$. We put $u^*(0) = u(+0 \mid \mathbf{z}^*)$.

We will assume that the stabilizer operates in the interval $[0, \tau_0]$, $\tau_0 \geqslant 0$. Then we construct a stabilizing control $u^*(t)$, $t \in [0, \tau_0[$ and the corresponding trajectory $\mathbf{z}^*(t)$, $t \in [0, \tau_0]$, of the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u^*(t)$, $\mathbf{x}(0) = \mathbf{z}^*$, $t \in [0, \tau_0]$. By (2.4), to construct the control $u^*(\tau)$ for $\tau \in T^+(\tau_0)$ we need to know the open-loop solutions $u(t \mid \mathbf{z}^*(\tau))$, $t \in [0, t^*(z^*(\tau))]$ of problems (2.1) when $\mathbf{z} = \mathbf{z}^*(\tau)$, $\tau \in T^+(\tau_0)$.

We will denote the parameters (2.6) constructed for $\mathbf{z} = \mathbf{z}^*(\tau)$ by

$$p(\tau),\ k_j(\tau),\ j = \overline{0, p(\tau)},\ t_j(\tau),\ j = \overline{1, p(\tau)},\ t^*(\tau),\ \rho(\tau) \tag{3.1}$$

We will call the function

$$\Delta_\tau(t) = -\mathbf{y}'(\tau)\mathbf{F}(t^*(\tau), t)\mathbf{b},\ t \in T(\tau) = T(\mathbf{z}^*(\tau)), \tag{3.2}$$

the optimal co-control of problem (2.1) for $\mathbf{z} = \mathbf{z}^*(\tau)$.

We will call the parameters (3.1) the defining elements of the open-loop solution of problem (2.1) when $\mathbf{z} = \mathbf{z}^*(\tau)$, since, knowing (3.1), we can easily construct the optimal open-loop equation

$$u(t|\mathbf{z}^*(\tau)) = k_j(\tau)\rho(\tau),\ t \in [t_j(\tau), t_{j+1}(\tau)[,\ j = \overline{0, p(\tau)} \tag{3.3}$$
$$t_0(\tau) = 0,\ t_{p(\tau)+1}(\tau) = t^*(\tau)$$

and the optimal co-control $\Delta_\tau(t)$, $t \in T(\tau)$, (3.2), along which the necessary condition for optimality (2.3) is satisfied.

We will henceforth assume that if there is a unique Lagrange-multiplier vector $\mathbf{y}(\tau)$ for $\mathbf{z} = \mathbf{z}^*(\tau)$ in problem (2.1) for the solution (3.3), then in the co-control corresponding to it the following relation is satisfied: for any $t_0 \in T(\tau)$ the following inequality follows from the equations $\Delta_\tau(t_0) = \partial\Delta_\tau(t_0)/\partial t = 0$

$$\partial^2\Delta_\tau(t_0)/\partial t^2 \neq 0 \tag{3.4}$$

We will initially assume that at the instant τ_0 the defining elements are such that

1. $t_1(\tau_0) > 0$;
2. $\operatorname{rank}(\mathbf{F}(t^*(\tau_0), t_j(\tau_0))b,\ j = \overline{1, p(\tau_0)}) = n - 1$;
3. $\partial\Delta_\tau(t)/\partial t|_{t=t_j(\tau_0)} \neq 0,\ j = \overline{1, p(\tau_0)}$.

It can then be shown that the defining elements (3.1) when $\tau \in T^+(\tau_0)$ can be found uniquely from the relations

$$\begin{aligned}
&p(\tau) = p,\ k_j(\tau) = (-1)^j k,\ j = \overline{1, p}\\
&\mathbf{\Phi}(\tau, k, t_j(\tau),\ j = \overline{1, p};\ t^*(\tau),\ \rho(\tau)) = 0\\
&q(t^*(\tau), t_j(\tau),\ \mathbf{y}(\tau)) = 0,\ j = \overline{1, p}\\
&q_*(\tau, t^*(\tau),\ \mathbf{y}(\tau), \rho(\tau)) = 0,\ q_0(k, p, \mathbf{y}(\tau), \rho(\tau)) = 0
\end{aligned} \tag{3.5}$$

where

$$\mathbf{\Phi}(\tau, k, t_j, j = \overline{1, p}, t^*, \rho) = \mathbf{F}(t^*, 0)\mathbf{z}^*(\tau) +$$
$$+\rho \sum_{j=0}^{p} (-1)^j k \int_{t_j}^{t_{j+1}} \mathbf{F}(t^*, t)\mathbf{b}dt,\ t_0 = 0,\ t_{p+1} = t^*;\ k = k_0(\tau_0),\ p = p(\tau_0)$$
$$q_*(\tau, t^*, \mathbf{y}, \rho) = \mathbf{y}'\mathbf{F}(t^*, 0)\mathbf{z}^*(\tau) - \rho,\ q(t^*, t, \mathbf{y}) = \mathbf{y}'\mathbf{F}(t^*, t)\mathbf{b}$$
$$q_0(k, p, \mathbf{y}, \rho) = \mathbf{y}'\mathbf{b}\rho(-1)^p k + 1$$

and from the initial conditions

$$t_j(\tau_0+0)=t_j(\tau_0),\quad j=\overline{1,p(\tau_0)},\quad \rho(\tau_0+0)=\rho(\tau_0),\quad \mathbf{y}(\tau_0+0)=\mathbf{y}(\tau_0)$$

In fact, we calculate the Jacobi matrix of Eqs (3.5)

$$\mathbf{G}=\left\| \begin{array}{cc|c} \multicolumn{2}{c|}{\mathbf{G}_1} & \mathbf{0} \\ \hline \mathbf{D} & \mathbf{f} & \mathbf{G}'_2 \\ \mathbf{d}' & & \end{array} \right\|$$

$$\mathbf{G}_2=\mathbf{G}_2(\tau,t^*,t_j,j=\overline{1,p},\ \rho)=$$

$$=(\mathbf{F}(t^*,t_j)\mathbf{b},\ j=\overline{1,p};\ \mathbf{b}\rho(-1)^p;\mathbf{F}(t^*,\tau)\mathbf{z}^*(\tau))$$

$$\mathbf{G}_1=\mathbf{G}_1(\tau,t^*,t_j,j=\overline{1,p},\ \rho,k)=\mathbf{G}_2\operatorname{diag}(\rho\alpha_j,j=\overline{1,p};\ 1;\ -1/\rho)$$

$$\alpha_j=2(-1)^{j-1}k,\quad j=\overline{1,p}$$

$$\mathbf{D}=\mathbf{D}(\tau,t^*,t_j,\ j=\overline{1,p},\ \mathbf{y})=\operatorname{diag}(-\partial\Delta_\tau(t_j)/\partial t,\ j=\overline{1,p};\ \mathbf{y}'\mathbf{b}(-1)^p)$$

$$\mathbf{f}=(\mathbf{y}'\mathbf{AF}(t^*,t_j)\mathbf{b},\ j=\overline{1,p},0,\ \mathbf{y}'\mathbf{AF}(t^*,\tau)\mathbf{z}^*(\tau))'$$

$$\mathbf{d}'=(d_j=0,\ j=\overline{1,p};\ -1)$$

When conditions 1–3 are satisfied the following inequality holds

$$\det\mathbf{G}=\det\mathbf{G}(\tau_0,t^*(\tau_0),\ k,t_j(\tau_0),\ j=\overline{1,p},\ \rho(\tau_0),\ \mathbf{y}(\tau_0))\neq 0$$

According to the theorem of implicit functions, when $\tau\in T^*(\tau_0)$ there are unique continuous functions (3.1) for which relations (3.5) are satisfied identically.

For $\tau\in T^+(\tau_0)$ the stabilizing equation is constructed using the rule

$$u^*(\tau)=\rho(\tau)k(\tau_0),\quad \tau\in T^+(\tau_0) \tag{3.6}$$

The governing elements (3.1) and the stabilizing control can be found from rules (3.5) and (3.6) so long as relations 1–3 hold.

4. AN ALGORITHM FOR CONSTRUCTING THE STABILIZING CONTROL. THE CASE WHEN RELATIONS 1–3 BREAK DOWN

Relations 1–3 may break down as a result of the fact that for a certain $\tau_1>\tau_0$ one of the following situations occurs

1. $t_1(\tau)\to 0$ as $\tau\to\tau_1-0$
2. $t_{j_0}(\tau)\to t_0,\ t_{j_0+1}(\tau)\to t_0$ as $\tau\to\tau_1-0$
3. $t_0\,\overline{\in}\,\{t_j(\tau_1-0),\ j=\overline{1,p},\},\ \Delta_{\tau_1}(t_0)=0$ exists.

We will assume that at the instant τ_1 only one of situations 1–3 can occur and on one index.

Consider case 1. We put $s_j=t_{j+1}(\tau_1-0), j=\overline{1,\bar{p}}, \bar{p}=p-1, \bar{k}=-k, s^*=t^*(\tau_1-0), \bar{\rho}=\rho(\tau_1-0)$. If

$$\operatorname{rank}(\mathbf{F}(s^*,s_j)\mathbf{b},\ j=\overline{1,\bar{p}})=n-1 \tag{4.1}$$

then, when $\tau\in T^+(\tau_1)$ the governing elements (3.1) and $u^*(\tau)$ are found from relations (3.5) and (3.6), where the signs p and k are replaced by $\bar{p}$, $\bar{k}$, and the initial conditions are

$$t_j(\tau_1+0)=s_j,\ j=\overline{1,\bar{p}};\ \rho(\tau_1+0)=\bar{\rho} \tag{4.2}$$

$$t^*(\tau_1+0)=s^*,\ \mathbf{y}(\tau_1+0)=\mathbf{y}(\tau_1-0)$$

Suppose condition (4.1) breaks down. In this case, when $\tau\in T^+(\tau_1)$ we have

$$t_j(\tau)=s_j-(\tau-\tau_1),\quad j=\overline{1,\bar p}$$

$$\rho(\tau)=\bar\rho,\quad t^*(\tau)=s^*-(\tau-\tau_1) \tag{4.3}$$

Hence, for $\tau\in T^+(\tau_1)$ the stabilizing control $u^*(\tau)$ is identical with the open-loop optimal control $u(\tau-\tau_1\,|\,\mathbf{z}^*(\tau_1))$, $\tau_1\geqslant\tau$, of problem (2.1) with $\mathbf{z}=\mathbf{z}^*(\tau_1)$

$$u^*(\tau)=u(\tau-\tau_1|\mathbf{z}^*(\tau_1)),\quad \tau\in T^+(\tau_1) \tag{4.4}$$

The stabilizer is "switched off" for a while. An estimator, which solves the problem

$$\begin{aligned}
&f_*(\tau)=\max_{y\in Y(\tau)}(\mathbf{y}'\mathbf{b}(-1)^{\bar p}\bar k)\\
&\mathbf{Y}(\tau)=\{\mathbf{y}\in R^n:\mathbf{y}'\mathbf{F}(s^*,\tau-\tau_1)\mathbf{z}^*(\tau)=\bar\rho\\
&\mathbf{y}'\mathbf{F}(s^*,s_j)\mathbf{b}=0,\quad j=\overline{1,\bar p}\\
&\mathbf{y}'\mathbf{F}(s^*,t)\mathbf{b}(-1)^j\bar k\leqslant 0,\quad t\in[s_j,s_{j+1}],\quad j=\overline{0,\bar p}\}\\
&s_0=\tau-\tau_1,\quad s_{\bar p+1}=s^*
\end{aligned} \tag{4.5}$$

for $\tau\in T^+(\tau_1)$ is "connected" instead of it. It can be shown that problem (4.5) always has a solution and, moreover, $f_*(\tau_1)>-1$, $f_*(\tau)\leqslant 0$, $\tau\geqslant\tau_1$.

We will denote the optimal plan of problem (4.5) by $y^*(\tau)$. It can be proved that

$$0<\bar\beta(\tau)=\mathbf{y}'(\tau_1-0)\mathbf{F}(s^*,\tau-\tau_1)\mathbf{z}^*(\tau)\leqslant\bar\rho,\quad \tau\in[\tau_1,s^*[$$

It follows from the last relation that if $f_*(\tau)>-1$, the vector

$$\mathbf{y}(\tau)=\lambda(\tau)\mathbf{y}(\tau_1-0)\bar\rho/\bar\beta(\tau)+(1-\lambda(\tau))\mathbf{y}^*(\tau)$$

where $0\leqslant\lambda(\tau)=:(1+f_*(\tau))/(\bar\rho/\bar\beta(\tau)+f_*(\tau))\leqslant 1$, is a vector of the Lagrange multipliers in problem (2.1) when $\mathbf{z}=\mathbf{z}^*(\tau)$, and along the solution of the conjugate system (2.2) and the control corresponding to it

$$\begin{aligned}
&u(t|z^*(\tau))=(-1)^j\bar\rho\bar k,\quad t\in[t_j(\tau),\ t_{j+1}(\tau)[\\
&j=\overline{0,\bar p};\quad t_0(\tau)=0,\quad t_{\bar p+1}(\tau)=t^*(\tau)
\end{aligned}$$

the necessary conditions of optimality (2.3) are satisfied and $t_j(\tau)$ $(j=\overline{1,\bar p})$, $t^*(\tau)$ are defined by (4.3).

Problem (4.5) is a problem of linear programming with a continuum of constraints. The algorithm for solving it in real time is analogous to that described previously [8]. The operation of the algorithm reduces to solving a special system of non-linear (governing) equations.

We act in accordance with rules (4.3) and (4.4) and solve problem (4.5) until, at a certain instant $\tau_2\geqslant\tau_1$, one of the following situations occurs: (a) $f_*(\tau_2)=0$; (b) $t_1(\tau_2)=s_1-(\tau_2-\tau_1)=0$; (c) $f_*(\tau_2)=-1$.

Consider case (a). It can be shown that when $\tau\in[\tau_2,s^*+\tau_2]$ the solution of problem (4.5) will have the form $\mathbf{y}^*(\tau)=\mathbf{y}^*(\tau_2)\gamma(\tau)$, $f_*(\tau)=0$, where $\gamma(\tau)=\bar\rho/\mathbf{y}^{*1}(\tau_2)\mathbf{F}(s^*,\tau-\tau_2)\mathbf{z}^*(\tau)>1$. Consequently, when $\tau>\tau_2$ there is no need to solve problem (4.5). When $\tau\in[\tau_2,s^*+\tau_2]$ the stabilizing control is constructed in accordance with rule (4.4). When $\tau>s^*+\tau_2$ we assume $u^*(\tau)\equiv 0$, $\tau>s^*+\tau_2$, since, by construction, $\mathbf{z}^*(s^*+\tau_2)=\mathbf{0}$.

In case (b) we assume $\bar s_j=t_{j+1}(\tau_2)$, $j=\overline{1,m}$, $m=\bar p-1$, $q=-\bar k$, $\bar s^*=t^*(\tau_2)$. For $\tau\in \underline{T^+}(\tau_2)$ we act in accordance with rules (4.3)–(4.5), where s_j, $j=\overline{1,\bar p}$, $\bar k$, τ_1 are replaced by $\bar s_j$, $j=\overline{1,m}$, q, τ_2 until, for a certain $\tau_3>\tau_2$ one of situations of the type (a)–(c) occurs.

Consider case (c). Since $f_*(\tau)>-1$, $\tau\in T^-(\tau_2)$, $f_*(\tau_2)$, $f_*(\tau_2)=-1$ we have $df_*(\tau_2)/d\tau\leqslant 0$. We will assume that $df_*(\tau_2)/d\tau<0$.

Consider the optimal plan $\mathbf{y}^*=\mathbf{y}^*(\tau_2)$ of problem (4.5). We put

$$\{t_j, j=\overline{1,l}\} = \{t \in [\tau_2 - \tau_1, s^*] \setminus \{s_j, j=\overline{1,\bar{p}}\}:$$
$$\mathbf{y}^{*\prime}\mathbf{F}(s^*, t)\mathbf{b} = 0\}$$

According to the criterion of optimality in problem (4.5) [9], numbers $\bar{\mu}_j \geqslant 0, j=\overline{1,l}$, $\mu_j, j=\overline{1,\bar{p}}$, μ_0 exist such that

$$\sum_{j=1}^{l} \bar{\mu}_j \mathbf{F}(s^*, t_j)\mathbf{b}\gamma_j + \sum_{j=1}^{\bar{p}} \mu_j \mathbf{F}(s^*, s_j)\mathbf{b} + \mu_0 \mathbf{F}(s^*, \tau_2 - \tau_1)\mathbf{z}^*(\tau) = \mathbf{b}(-1)^{\bar{p}}\bar{k}$$

where $\gamma_j = (-1)^i\bar{k}$, if $t_j \in]s_i, s_{i+1}[$.

We will assume that the solution of problem (4.5) when $\tau = \tau_2$ is non-degenerate, i.e. $\bar{\mu}_j \neq 0, j = \overline{1,l}$, $\operatorname{rank}(\mathbf{F}(s^*, t_j)\mathbf{b}, j = \overline{1,l}) = l$. To simplify our further calculations we will assume that $(\tau_2 - \tau_1) \,\overline{\in}\, \{t_j, j=\overline{1,l}\}$.

By virtue of assumption (3.4) we have

$$\mathbf{y}^{*\prime}\mathbf{F}(s^*, s_j)\mathbf{A}\mathbf{b} \neq 0,\ j=\overline{1,\bar{p}},\ \mathbf{y}^{*\prime}\mathbf{F}(s^*, t_j)\mathbf{A}^2\mathbf{b} \neq 0,\ j=\overline{1,l}$$

We put $m = \bar{p} + 2l$, $\{\bar{s}_j, j=\overline{1,m}\} + \{s_j, j=\overline{1,\bar{p}}, s_{1j} = t_j, s_{2j} = t_j, j=\overline{1,l}\}$, $\bar{s}_j \leqslant \bar{s}_{j+1}, j=\overline{1,m-1}$; $q = k$.

Taking the above assumptions into account it can be shown that when $\tau \in T^+(\tau_2)$ the governing elements (3.1) and the stabilizing control $u^*(\tau)$ are found by rules (3.5) and (3.6), where p and k are replaced by m and q, starting from the initial conditions

$$t_j(\tau_2+0) = \bar{s}_j - (\tau_2 - \tau_1),\ j=\overline{1,m};\ \rho(\tau_2+0) = \bar{\rho}$$
$$t^*(\tau_2+0) = s^* - (\tau_2-\tau_1),\ \mathbf{y}(\tau_2+0) = \mathbf{y}^*$$

We act in accordance with rules (3.5) and (3.6) until one of the situations 1–3 occurs.

Consider situation 2. Clearly, $\dot{t}_{j0+1}(\tau_1) - \dot{t}_{j0}(\tau_1) \leqslant 0$. We will assume that $\dot{t}_{j0+1}(\tau_1) - \dot{t}_{j0}(\tau_1) < 0$. We put

$$s_j = t_j(\tau_1 - 0),\ j=\overline{1,j_0-1},\ \ s_j = t_{j+2}(\tau_1+0),\ \ j=\overline{j_0,\bar{p}},\ \ \bar{p} = p-2$$
$$\bar{k} = k,\ s^* = t^*(\tau_1 - 0),\ \bar{\rho} = \rho(\tau_1 - 0)$$

We then act in accordance with the rules of case 1.

Consider case 3. It can be shown that $t_0 \neq 0$, $t_0 \neq t^*(\tau_1)$. Consequently, $t_0 \in \operatorname{int} T(\tau_1)$. Suppose, to fix our ideas, that $t_0 \in]t_{j_*}(\tau_1), t_{j_*+1}(\tau_1)[$. It is obvious that $d\mathbf{y}'(\tau_1)\mathbf{F}(t^*(\tau_1), t_0)\mathbf{b}(-1)^{j_*}k/d\tau \geqslant 0$. We will assume that $d\mathbf{y}'(\tau_1)\mathbf{F}(t^*(\tau_1), t_0)\mathbf{b}(-1)^{j_*}k/d\tau > 0$.

We put $s_j = t_j(\tau_1 - 0), j=\overline{1,j_*}$, $s_{j_*} = s_{j_*+1} = t_0$, $s_j = t_{j-2}(\tau_1 - 0), j=\overline{j_*+2,\bar{p}}$; $\bar{p} = p+2$, $\bar{k} = k$, $\bar{\rho} = \rho(\tau_1 - 0)$, $s^* = t^*(\tau_1 - 0)$. We then act in accordance with the rules of case 1. Note that in this situation relations (4.1) will always be satisfied.

We described above an algorithm for constructing the function $u^*(\tau)$, $\tau \geqslant 0$ on the assumption that for any $\tau \geqslant 0$ the solution of problem (2.1), $\mathbf{z} = \mathbf{z}^*(\tau)$, constructed from the governing elements (3.1), is a unique solution of problem (2.1). In this case the functions $\rho(\tau)$, $t^*(\tau)$, $\tau \geqslant 0$ will be continuous.

In the general case, a situation is possible in which, for certain $\bar{\tau}$ in problem (2.1) with $\mathbf{z} = \mathbf{z}^*(\bar{\tau})$ in addition to the solution

$$u(t|\mathbf{z}^*(\bar{\tau})) = \rho(-1)^j k,\ t \in [t_j(\bar{\tau}),\ t_{j+1}(\bar{\tau})[$$
$$j=\overline{0,p},\ t_0(\bar{\tau}) = 0,\ t_{p+1}(\bar{\tau}) = t^*(\bar{\tau}),\ \rho(\bar{\tau}) = \rho \tag{4.6}$$

corresponding to the governing elements constructed from the rules described above, there will be one other solution

$$u(t|\mathbf{z}^*(\bar{\tau})) = \rho^{(*)}(-1)^j k^{(*)},\ t \in [t_j^{(*)}(\bar{\tau}),\ t_{j+1}^{(*)}(\bar{\tau})[$$
$$j=\overline{0,p^{(*)}};\ t_0^{(*)}(\bar{\tau}) = 0,\ t_{p^*+1}^{(*)}(\bar{\tau}) = t^{(*)^*}(\bar{\tau}),\ \rho^{(*)}(\bar{\tau}) = \rho^{(*)} \tag{4.7}$$

with a smaller value of the parameter $\rho^{(*)}$: $\rho^{(*)} < \rho$. (It can be shown that no other solution with a value of the parameter $\rho^{(*)} > \rho$ can occur.)

In this case we put $s_j = t^{(*)}{}_j(\bar{\tau}), j = \overline{1, \bar{p}}, \bar{p} = p^{(*)}, s^{(*)} = t^{(*)*}(\bar{\tau}), \bar{\rho} = \rho^{(*)}(\bar{\tau}), \bar{k} = k^{(*)}$. We then act in accordance with the rules of case 1, replacing τ_1 by τ^{-}. Note that the functions $t^*(\tau)$, $\rho(\tau)$ are discontinuous at the point $\tau = \bar{\tau}$.

5. EXAMPLE

As an illustration we will consider the problem of stabilizing a mathematical pendulum in the upper unstable equilibrium position by a moment applied to it at the axis of suspension. This moment is generated by a slave mechanism, which is an integrating circuit. The behaviour of the slave mechanism is in turn regulated by a certain controlling force u [4].

The equation of the perturbed motion has the form

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = \sin x_1 + x_3, \quad \dot{x}_3 = u_2$$

where $x_1 = \varphi$ is the angle of deflection of the pendulum for the vertical and $x_1 = \dot{\varphi}$, x_3 is the moment applied to the pendulum.

We will write the equation of the first approximation

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_1 + x_3, \quad \dot{x}_3 = u \tag{5.1}$$

To stabilize system (5.1) when $\tau > 0$ we must solve the problem

$$t^* + 5\rho \to \min, \quad \dot{x}_1 = x_2, \quad \dot{x}_2 = x_1 + x_3, \quad \dot{x}_3 = u \tag{5.2}$$

$$\mathbf{x}(0) = \mathbf{z}^*(\tau), \quad x_1(t^*) = x_2(t^*) = x_3(t^*) = 0, \quad |u(t)| \leqslant \rho, \quad t \in [0, t^*]$$

where $\mathbf{z}^*(\tau) = (z_i^*(\tau), i = 1, 2, 3)$ is the state of system (5.1) at the instant τ, reached due to the action of the control $u^*(t)$, $t \in [0, \tau[$ generated. The control $u^*(t)$ is constructed in accordance with the rules described in Sections 3 and 4.

In this example we took $\mathbf{z}^* = (1.27; -1.67; 0)$ as the initial state. For $\mathbf{x}(0) = \mathbf{z}^*$ the function $f(t^*) = t^* + 5\rho(t^*)$ (1.3) and (1.4) has two local minima (Fig. 1): (1) at the point $t^*_{(1)} = 1.9947$ a non-smooth minimum with $\rho(t^*_{(1)}) = 1.0042$, $f(t^*_{(1)}) = 7.016$; (2) at the point $t^*_{(2)} = 3.294$ a smooth minimum with $\rho(t^*_2) = 0.782$, $f(t^*_{(2)}) = 7.204$. When $t^* = t^*_{(1)}$ the optimal control of problem (1.4) has a single reversing point $t_{1(2)} = 0.9974$ when $t^* = t^*_{(2)}$ the optimal control of problem (1.4) has two reversing points $t_{1(2)} = 1.264$, $t_{2(2)} = 2.912$. Since $f(t^*_{(1)}) < f(t^*_{(2)})$, the solution of

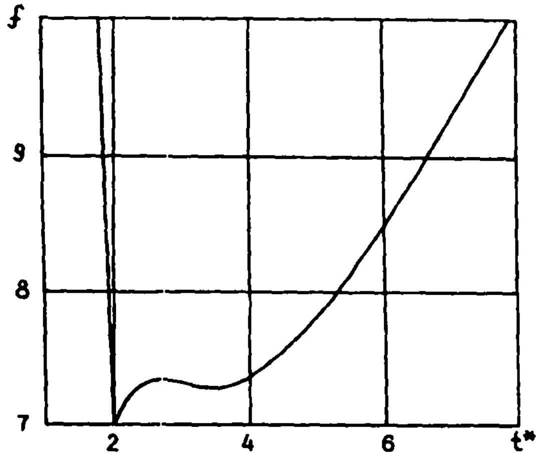

Fig. 1.

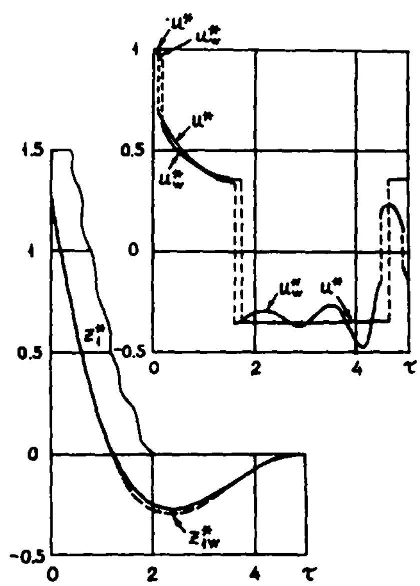

Fig. 2.

problem (5.2) when $\tau = 0$, $\mathbf{z}^*(0) = \mathbf{z}^*$ has the form

$$
\begin{aligned}
&u(t|\mathbf{z}^*) = \rho(\mathbf{z}^*), \quad t \in [0, t_1(\mathbf{z}^*)[\\
&u(t|\mathbf{z}^*) = -\rho(\mathbf{z}^*), \quad t \in [t_1(\mathbf{z}^*), \ t^*(\mathbf{z}^*)] \\
&t^*(\mathbf{z}^*) = t_1^*, \ \rho(\mathbf{z}^*) = \rho(t_{(1)}^*), \ t_1(\mathbf{z}^*) = t_{1(1)}
\end{aligned}
\tag{5.3}
$$

The process of stabilization begins with solution (2.3). Here, when $0 < \tau < 0.12$, to construct the control $u^*(\tau)$ we use rules (4.3) and (4.4). At the instant $\tau = \bar{\tau} = 0.12$ we have the situation described at the end of Section 4, when, with $\tau = \bar{\tau}$, problem (5.2) has two solutions: (1) solution (4.6), where $p = 1$, $k = 1$, $t_1(\bar{\tau}) = t_{1(1)} - \bar{\tau}$, $t^*(\bar{\tau}) = t^*_{(1)} - \bar{\tau}$, $\rho = \rho(t^*_{(1)})$; (2) solution (4.7), where $p^{(*)} = 2$, $k^{(*)} = 1$, $t^*_{(1)}(\bar{\tau}) = 1.35 - \bar{\tau}$, $t^*_{(2)}(\bar{\tau}) = 3.18 - \bar{\tau}$, $t^{(*)^*}(\bar{\tau}) = 3.62 - \bar{\tau}$, $\rho^{(*)} = 0.679$. Since $\rho^{(*)} < \rho$, then, in accordance with the algorithm, we assume

$$
s_j = t_j^{(*)}(\bar{\tau}), \ j = 1, 2; \ s^* = t^{(*)^*}(\bar{\tau}), \ \bar{\rho} = \rho^{(*)}, \ \bar{k} = k^{(*)}, \bar{p} = 2
\tag{5.4}
$$

and we act in accordance with the rules of case 1, assuming $\tau_1 = \bar{\tau}$. We have (4.1) for the parameters (5.4). Consequently, when $\tau \in T^+(\bar{\tau})$ the control $u^*(\tau)$ is constructed in accordance with the rules (3.5) and (3.6), where p and k are replaced by $\bar{p}$, $\bar{k}$ and initial conditions (4.2).

The rules (3.5) and (3.6) are used up to the instant $\tau = \bar{\tau}_1 = 1.69$. At the instant $\bar{\tau}_1$ we have $t_1(\bar{\tau}_1) = 0$, $t_2(\bar{\tau}_1) = 4.62 - \bar{\tau}_1$, $t^*(\bar{\tau}_1) = 5.2 - \bar{\tau}_1$, i.e. we obtain situation 1. For $s_1 = t_2(\bar{\tau}_1)$, $s^* = t^*(\bar{\tau}_1)$ and $p = 1$ relations (4.1) break down, and hence when $t \in T^+(\bar{\tau}_1)$ we act according to the rules (4.3)–(4.5). Since $f_*(\bar{\tau}_1) = 0$, we have case *a* when $\tau = \bar{\tau}_1$.

According to the algorithm for $1.69 \leqslant \tau \leqslant 5.2$ the stabilizing control $u^*(\tau)$ is constructed according to rule (4.4). For $\tau > 5.2$ we assume $u^*(\tau) = 0$ as $x^*(5.2) = 0$. The stabilization is completed.

We will consider the process of stabilization of system (5.1) for constantly acting unknown perturbations $w(t)$, $t \geqslant 0$, under the influence of which system (5.1) takes the form

$$
\dot{x}_1 = x_2, \ \dot{x}_2 = x_1 + x_3 + w(t), \ \dot{x}_3 = u
\tag{5.5}
$$

To stabilize (5.5) when $\tau > 0$ we solve the problem

$$
\begin{aligned}
&t^* + 5\rho \to \min, \ \dot{x}_1 = x_2, \dot{x}_2 = x_1 + x_3, \ \dot{x}_3 = u \\
&\mathbf{x}(0) = \mathbf{z}^*_w(\tau), \ x_1(t^*) = x_2(t^*) = x_3(t^*) = 0, \ |u(t)| \leqslant \rho, \ t \in [0, t^*]
\end{aligned}
$$

where $\mathbf{z}^*_w(\tau) = (z^*_{wi}(\tau), i = 1, 2, 3)$ is the state of system (5.5) at the instant τ, reached under the action of the generated control $u^*_w(t)$, $t \in [0, \tau[$, and the perturbation $w(t)$, $t \in [0, \tau[$. The control $u^*_w(t)$ is constructed using rules similar to those described in Sections 3 and 4 and [7]. In the example we took $w(t) = 0.1\cos 5t$, $t \geqslant 0$ as the perturbation.

In Fig. 2 we show the behaviour of $\mathbf{z}^*{}_1(\tau)$ when $0 \leqslant \tau \leqslant 5$ due to the action of the generated control $u^*(\tau)$ (the continuous curve), and the behaviour of $\mathbf{z}^*{}_{w1}(\tau)$ under the action of the generated control $u^*_w(\tau)$ and the perturbation $w(\tau) = 0.1\cos 5\tau$ (the dashed curve), and also $u^*(\tau)$ and $u^*_w(\tau)$.

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REFERENCES

1. ATANS M. and FALB P. L., *Optimal Control*. Mashinostroyeniye, Moscow, 1968.
2. LETOV A. M., Analytical construction of controllers. *Avtomatika i Telemekhanika* **21**, 4–6, 436–441, 561–568, 661–665, 1960; **22**, 4, 425–435, 1961; **23**, 11, 1405–1413, 1962.
3. KALMAN R. E., The general theory of control systems. In *Proc. 1st Congress of the IFAC*, Vol. 2, pp. 521–547, Izd. Akad. Nauk SSSR, 1961.
4. MALKIN I. G., *Theory of the Stability of Motion*. Nauka, Moscow, 1966.
5. MAYNE D. Q. and MICHALSKA H., Receding horizon control of nonlinear systems. *IEEE Trans. Aut. Control* **35**, 7, 814–824, 1990.
6. KWON W. H. and PEARSON A. E., A modified quadratic cost problem and feedback stabilization of a linear system. *IEEE Trans. Aut. Control* **22**, 5, 838–842, 1977.
7. GABASOV R., KIRILLOVA F. M. and KOSTYUKOVA O. I., Methods of stabilizing dynamic systems. *Izv. Ross. Akad. Nauk. Tekh. Kibernetika* 3, 67–77, 1994.
8. GABASOV R., KIRILLOVA F. M. and KOSTYUKOVA O. I., Optimal positional observation of linear systems. *Dokl. Ross. Akad. Nauk* **339**, 4, 461–464, 1994.
9. KOSTYUKOVA O. I., Superbasis plans of the linear extremal problem with a continuum of constraints. *Dokl. Akad. Nauk BSSR* **33**, 8, 687–689, 1989.

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